STUDENTS’ UNDERSTANDING OF THE GENERALITY OF ALGEBRAIC PROOFS AND OPERATIVE PROOF IN SECONDARY SCHOOL MATHEMATICS EDUCATION

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In this paper we report our teaching experiment in which students in a lower secondary school undertook number investigations (ANNA numbers) with manipulative objects (operative proofs) before they constructed algebraic proofs. After examining students’ activities in lessons, their coursework and post test questions, we conclude that the idea of generality of proof emerged when students engaged in proving with operative proofs. This activity can also be a starting point to make new conjectures and prove them.

1. INTRODUCTION

In Japanese mathematics education, previous research evidence suggests that even though students could ‘construct’ a formal proof using algebra to verify mathematical statements, they still also considered that using numerical examples for this purpose would be equally valid (e.g. Kunimune et al., 2009). In order to improve this situation, learning environments should be designed which highlight the value of algebraic proof to students. In our study, we consider the use of operative proof, proposed by Wittmann (1996), can enhance learners’ understanding of algebraic proofs, in particular the generality of proofs.

The focus of this paper is proof and proving in the teaching of algebra in lower secondary schools which is one of the themes of TSG9, and we shall report findings from our teaching experiment in which students undertook number investigation tasks. In particular the students used manipulative objects to reason and justify their findings before they constructed algebraic proofs. The purpose of this paper is to consider the following research questions: ‘What learning processes do students experience?’ and ‘How did the understanding of the generality of proof emerge when students study operative proofs?’

2. OPERATIVE PROOFS IN ALGEBRAIC PROOF

2.1. Proof and Proving in schools

Existing research studies suggest that learners do not always use an algebraic proof to prove a statement (e.g. Harel and Sowder’s proof scheme, 1998). Instead, a learner might a) just verify with a few examples, b) start checking the statement’s generality with more examples, or c) use counters to check and verify its generality. In terms of Balacheff’s taxonomy
(1988), a), b) and c) are all classified as belonging to the ‘pragmatic proof’ category. a) is characterised by ‘naïve empiricism’, b) by the use of a ‘crucial experiment’ and c) by extending this to ‘generic examples’.

To define what is meant by proof in the school mathematics context, the notion of proof by Stylianides (2007) is useful, whereby a proof is regarded as a mathematical argument with a ‘set of accepted statements’, ‘modes of argumentation’ and ‘modes of argument representation’ (p. 291). From this point of view, it is considered that the third level described above, ‘c)’, for example the use of counters to check and verify the generality of mathematical statements (i.e. the generic examples) can be considered as a ‘proof’ (a detailed example is shown in 2.3. See also Stylianides and Stylianides, 2008; Komatsu, 2010).

2.2. Wittmann’s operative proof

Recent studies have shown the importance of proving activities with visual and manipulative objects, not only in the early but also advanced stages of the teaching and learning of number and algebra. For example, based on Balacheff’s taxonomy, Miyazaki (2000) examined various types of proofs, and concluded proof with manipulations of concrete objects can bridge the gap between empirical and formal mathematical arguments. Komatsu (2010) closely studied how G5 children utilized manipulative objects to pose and refine conjectures and then produce mathematical argumentations to form a proof.

Our investigation is along similar lines, i.e. using visual and manipulative representation to facilitate students’ proving process in algebra in lower secondary schools, and we particular adopt the idea of ‘Operative proof’ proposed by Wittmann (1996). The theoretical basis is Piaget’s theory of genetic epistemology and reflective abstraction, “drawn from the general coordination of actions or of operations” (Piaget, 1980, pp. 89-97), and in particular Wittmann attempted to apply the idea of ‘operation’ to the teaching and learning of proofs in various contexts. He suggests operative proof has the following two characteristics: it is “(1) integrated in the exploration of a mathematical context; and (2) based on the effects of operations exerted thereby on meaningfully represented mathematical objects.” (p.6) A key idea is to present abstract mathematical concepts with suitable representations and to make them possible to manipulate. By such representations, it is possible for various levels of learners to reflect on operations with the representations and this makes it possible for students to experience, for example, exploration, reasoning and communication about mathematical patterns.

2.3. Example of Operative proof: ANNA numbers

Let us take an example, ‘ANNA numbers’ which are introduced by Wittmann (1996) to illustrate the operative proof which is the focus of our study. ANNA and NAAN numbers are pairs of four digits numbers from 0-9, such as (4334, 3443), (5335, 3553) etc. and the differences of these numbers are always ‘891 x (A-N); A>N’, i.e. 4334-3443=891x(4-3), 5335-3553=1782=891x(5-3).

To ‘prove’ this, the following algebraic proof is considered: ‘1000A+100N+10N+A-(1000N+100A+10A+N)=891A-891N=891(A-N); A>N, Whereas this is a valid algebraic proof, the following approach can also be considered. First, by using
Last names of authors in order as on the paper

In terms of the notion of proof discussed in the previous section, this ‘proof’ uses ‘manipulation’ in the form of the place value table ‘+1000-100-10+1=891’ which can be used as an accepted statement. The mode of argument is general as this ‘operation’ can be applied to any ANNA/NAAN numbers with any differences between A and N, and the mode of argument representation is pictorial.

3. METHODOLOGY

3.1. Study context

The issues to be addressed are whether this kind of ‘proof’ functions for students to reason and justify mathematical statements and is useful to develop ideas of ‘proof’ in lower secondary schools. To investigate this, we designed and implemented a sequence of lessons with ANNA numbers for G8 students in lower secondary schools in Japan (14 yrs old). To design the sequence, we borrowed and modified the ANNA number tasks in Das Zahlenbuch edited by Wittmann and Müller, which is widely used in German primary schools. In this book, children in G4 investigate the patterns in ANNA numbers with place value tables. By borrowing ideas from this textbook, we designed the following four tasks to enable different stages of learning for lower secondary school students in which operative proof is used to bridge the gap between empirical investigations and formal proof in algebra: (1) Calculate the differences between ANNA numbers; (2) Examine what patterns can be observed in general; (3) Reason the pattern with counters and a place value table (operative proof); (4) Prove the pattern of ANNA number by using algebra.

3.2. Data in this study

The data collected in our study are derived from the lessons which were implemented in 4 G8 classes. These are referred to as A, B, C and D, and the number of students is 40, 40, 41, and 41 respectively, and boys:girls=1:1. All classes belonged to a University-attached lower secondary school in Japan. The standard of the students is relatively high because this is a somewhat selective school with a particularly highly qualified and experienced teaching staff, but several students still have difficulties in mathematics. The students have in general established good relationships among their peers and enjoy learning mathematics.
collaboratively and presenting their ideas. The teachers have more than 10 years of teaching experience of mathematics.

We conducted an experimental research programme with 4 lessons in algebra units, described in the following table 1.

<table>
<thead>
<tr>
<th>Lesson</th>
<th>Experimental group (Class B, D)</th>
<th>Control group (Class A and C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>• Investigations with ANNA numbers</td>
<td>• Learning how to explain with algebra based on textbooks (odd number + odd number = even numbers)</td>
</tr>
<tr>
<td></td>
<td>• Operative proof with place value tables (Cases of A-N=1)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>• Operative proof with place value tables (general cases)</td>
<td>• Consolidation with examples from textbooks</td>
</tr>
<tr>
<td></td>
<td>• Learning how to explain with algebra</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>• Consolidation with examples from textbooks (odd number + odd number = even numbers)</td>
<td>• ANNA numbers with algebra</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>Unit test</td>
</tr>
</tbody>
</table>

As we can see above, while both groups study almost the same mathematical content, the experimental group was introduced to the operative proof with the place value tables when they studied ANNA numbers, while the control group were just taught relevant algebraic proofs. The experimental group of students was also taught how to explain mathematical statements algebraically. All students were told this was a part of a research project and agreed to participate.

The main data collected and analysed are the lesson transcriptions. In addition to these, both groups were given a mathematical investigation task with ANNA numbers, and students’ coursework reports were also examined to see their degree of understanding of the topic and general approaches in proving.

4. FINDINGS

4.1. Students’ use of the place value table in ANNA number investigation

Students first undertook the case of A-N=1. They soon noticed that the difference would be always 891, and the place value table was introduced as a tool to investigate this. First, they constructed a representation of 3223 and then tried to represent 3223-2332 by taking 2 counters from the Thousands column, 3 from the Hundreds column, and so on. This exercise failed to give the students an answer, which created some confusion among the students. The teacher therefore gave an example which acted as a hint to the students that they must start with representing the smaller number in the place value table (T=teacher and S=students).

T: OK, what does this represent? [2332 with the place value table]
S1: 2332?
T: Yes, it is 2332. Now, S1, how can you represent 3223 from here?
S1: [gesture: 1 counter from hundreds to left, and 1 from tens to right]
T: Yes, I think you have a good imagination. OK, can you actually do it in front of everybody?
S1: [S1 moved to 1 counter from hundreds to thousands, and 1 from tens to ones]
T: OK, thank you, and, what is this [newly created table]
Ss: 3223.
T: OK? This is my hint for you.

This illustration worked well and gave the students direction as to how to manipulate the place value table to explain 3223-2332=891. Soon after, some of them started making sense of the significance of this operation.

S2: Aha! I see.
S3: When we move 1 counter from tens to ones, this means -9.
S2: That’s it!
S3: And then, from hundreds to thousands…
S2&S3: +900, and +900-1=891?
S4: 900-9!
Ss: 900-9=891!
S5: Wow!
S2: We did it!
S3: Yes!

As we can see, this intervention by the teacher worked well and the students started to grasp how this operation could be used to explain ANNA numbers’ properties. In the second lesson, the students also continued to investigate the other cases of ANNA numbers and again they used the operative proof well to explain and justify why the difference is always (A-N)x891.

4.2. Emergence of the generality of proof

In the third lesson, after they studied ANNA numbers with the place value table, the students in our experimental group also undertook a problem ‘What can we say when we add an odd number and another odd number?’, which is one of the commonly used problems in textbooks as an introductory problem in the unit. In fact, the control group students studied this problem in their first lesson with the following progression; first they learnt how to represent even/odd numbers by using letters, and then tried to prove the problem algebraically, which is commonly seen in usual lessons. On the other hand, the student in our experimental group experienced this lesson in a different way. In this lesson, some students started utilising operative proofs to explain why odd+odd=even, and this unexpectedly promoted discussion about generality of algebraic proofs, which was not
observed in the control group students. For example, in class D, the following interactions were observed.

T: OK, please explain your method, S1.
S1: If you visualise odd numbers, then it will be like this, and we have 1 block sticking out... Are you following me?
Ss: Yes.
S1: And then, this one [the sticking out block in the left] and this one [the sticking out block in the right] are combined, and we have this shape [a rectangle] and we have an even number? Do you understand me, everyone?
T: What do you think about this?
Ss: [silence]
T: No? Nothing? I thought you had an idea, because...
S1: Well whatever numbers we have, we ALWAYS have, here, 1 block sticking out...
T: I was waiting for you to say something like that... OK, S2, it seems you are not convinced?
S2: Yes, yes, I am convinced, but ...
T: But I have a question, for example, is it OK for other numbers? We are only consider the case of 7+5...
S2: I thought, what S1 wants to say is no matter how big the numbers we have are, it will still be like this, I mean, 1 sticking out block moves and the other one as well, and I think it is OK.

This interaction suggests that ideas of generality emerged as an unavoidable activity in students’ explanations with operative proofs. A similar discussion was also observed in class B. In this class, when the teacher questioned about the generality of operative proof, a student expressed the following comment without much instruction from the teacher.

S3: Uh, there are many numbers, not just 3 and 5, so I used [algebraic] letters to cover all numbers.

This indicates that activities with operative proofs and subsequent discussions triggered students to consider and appreciate the generality of proof in algebraic proofs. This is a significant step for Japanese students where many students cannot see the generality of formal proof in algebra (e.g. Kunimune, et al, 2009).

4.3. Students’ answers in the unit test and coursework reports

In the unit test, students undertook the following problem: AANN and NNAA numbers are pairs of four digits numbers from 0-9, such as (4433, 3344), (5533, 3355) etc. ‘Find patterns of the differences between

Student’s answers in the unit test
AANN and NNAA number and prove your findings’. Fig. 2 below shows that one student utilized operative proofs to represent the difference and then constructed a formal proof.

Also, even the students who failed to answer these questions correctly still attempted to construct operative proofs, while in the control group these students often did not know where to start their investigations and proof, as shown in the table 2 and table 3 below.

Table 2: Finding patterns

<table>
<thead>
<tr>
<th></th>
<th>Experimental group</th>
<th>Control group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct answer</td>
<td>77 (96.3%)</td>
<td>69 (87.3%)</td>
</tr>
<tr>
<td>Wrong answer</td>
<td>3 (3.8%)</td>
<td>8 (10.1%)</td>
</tr>
<tr>
<td>Non-response</td>
<td>0 (0.0%)</td>
<td>2 (2.5%)</td>
</tr>
</tbody>
</table>

Table 3: Proving the findings

<table>
<thead>
<tr>
<th></th>
<th>Experimental group</th>
<th>Control group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct answer</td>
<td>71 (88.8%)</td>
<td>64 (81.0%)</td>
</tr>
<tr>
<td>Wrong answer</td>
<td>7 (8.8%)</td>
<td>12 (15.2%)</td>
</tr>
<tr>
<td>Non-response</td>
<td>2 (2.5%)</td>
<td>3 (3.8%)</td>
</tr>
</tbody>
</table>

As a tentative conclusion, these results indicate that operative proofs are effective in offering students an intellectual tool to approach challenging problems.

The students were asked to investigate ANNA numbers further in their coursework whereby they were required to create new numbers and patterns, make conjectures with several examples and then attempt a proof. In the experimental group, students utilized operative proofs to explore their own investigations. For example, fig. 3 right shows that one student created her own numbers ABCD numbers and tried to identify patterns related to the difference between ABCD and DCBA numbers (e.g. 7654-4567=3087, 9876-6789=3087 and so on).
5. DISCUSSION

Now, we shall discuss the use of operative proofs in lower secondary schools in relation to our research questions: ‘What learning processes do students experience?’ and ‘How did the understanding of the generality of proof emerge when students study operative proofs?’. Existing literature and theoretical discussions suggest that operative proof can provide us with opportunities for productive mathematical activities, and our findings also indicate some positive outcomes. Activities with operative proof can be very useful for secondary school students. By using place value tables, our students successfully explained and justified the properties of ANNA numbers. Also, the students in our experiment clearly saw that their operation with the place value table could cover any case of either ANNA numbers, i.e. they recognised the generality of this operative proof. The following developmental process of understanding can be gleaned from our data: first students try various operations with manipulative objects; second, they grasp the meanings of their operations and use them to justify mathematical statements; third, they start noticing the generality of operations; and finally, they start appreciating algebraic proofs.

In the process of students’ exploration of their mathematical activities, operative proofs can be a useful starting point to make new conjectures and prove them, as can be observed in students’ coursework investigations. Nevertheless, as we have seen, the students first conducted the operation the wrong way round and this caused some confusion among students. To focus their thinking on how to ‘prove’ their findings, it is necessary for teachers to give guidance as to how to use the place value tables to examine the properties of ANNA numbers.

In this paper, we particularly focused on how the idea of generality of proofs emerged from learning with operative proofs. In addition to this important issue, we would like to continue to investigate how operative proof can be used effectively to develop students’ understanding of proof in algebra in future research.

References


Komatsu, K. (2010). Counter-examples for refinement of conjectures and proofs in primary


