It is a great pleasure for me, indeed, to have the opportunity to present you some results of our recent research. I would like to thank Professor Yamamoto for the kind invitation to the university of Kumamoto. In the years 2004 and 2006 we enjoyed visits of Professor Yamamoto and his team to our university and appreciated two wonderful contributions to our annual symposium: one workshop about kirigami in 2004, and a second one this year about a highly original teaching experiment on number pyramids. Both workshops were extremely well received by the participating teachers. Frankly speaking I have been very curious to experience the atmosphere at Kumamoto University in which these substantial learning environments have been created, and I am very happy to be here today. Thank you once more.

The topic of my talk is “early mathematical education”. In Germany this field of research has found the strong interest of politicians after the “poor” results of the TIMMS and PISA studies had shocked the German public which did not take time to check if these studies are reliable or not.

In our project “mathe 2000” early mathematical education has been an important part from the very inception of this project 20 years ago as we have always been addressing the whole spectrum of mathematical learning. In doing so we have based our work on a certain view of mathematics, namely as the science of patterns.

From the very beginning we have been aware that also in education the early stages are the most important ones as expressed by a Chinese wise man more than 2000 years ago.

I should confess that recently my interest in early education has been stimulated also by my experiences as a grandfather.

This picture shows the family of our oldest son Christoph. The picture was taken one year ago on Isabelle’s first school day. Behind Isabelle you see my wife and our daughter-in-law. Isabelle’s brothers are twins: they will enter school next year. I will come back to grandma and Isabelle later in my talk.
This picture shows our second son Hans and our youngest grandchild Simon who is one year old and has just learned to walk.

There are several good reasons for early mathematical education.

First of all we must recognize that children can learn a lot of mathematics before they enter school, and most children with an appropriate social environment acquire surprising competencies in this period of time.

This graphic shows the percentages with which school beginners solve certain task with numbers:

- Over 90% are able to name the number signs of numbers up to 10.
- Over 60% are able to count backwards.
- Over 80% are able to mark a given number of circles in a 4x5 array.
- Over 60% are able to add small numbers.
- 50% of the school beginners are able to subtract small numbers.

This study was first done by Marja van den Heuvel in the Netherlands in the early nineties and then repeated in several countries with similar results. A second part of her study concerned the predictions of the percentages by experts, that is, by student teachers, teachers, school inspectors and maths educators: it turned out that the predictions of experts were far lower than the real percentages. That is experts tend to underestimate children dramatically. For example, the prediction for the last item was less than 10%.

However, in one of the follow up studies a Swiss team separated the data for different classes and found enormous differences. In one class 80% of all tasks were solved, in other classes only 20%. This points to the enormous influence of the social environment on children’s development and gives two more reasons for early mathematical education:

- Children who grow up in an unfavourable environment need systematic support far before their school education. Early education allows also for identifying learning difficulties and to start remedial measures at a young age.
- In summary: There are convincing reasons for investing consciously in early education both intellectually, emotionally and last not least financially.

What are the objectives and contents of early mathematical education?

In Germany there is a wide spectrum of views. The 16 states of the federal republic of Germany are independent as far as education is concerned. There are states like Bremen
where there is literally no early education, and on the other side of the spectrum is Bavaria with an ambitious and elaborated program.

**T 11** As you see there are three subject areas: numbers and magnitudes, geometry and logic. For each of these areas certain objectives are specified.

**T 12** Within German pedagogy and maths education the approaches are similarly diverse: Some believe that it is best to integrate mathematical tasks into the daily life of children. This so-called “situative approach” is propagated by pedagogues who specialize in early education and very popular among kindergarten nurses.

In the bookshops of German department stores you will find materials which are like workbooks in grade 1.

A colleague from Freiburg, Gerhard Preiß, developed a big program which he called “numberland”. This is an installation in a room with number houses connected by a number road. The houses are inhabited by numbers which behave like living beings, speak and do all kinds of things. Also there is a magician, a number devil and a number fairy. It is a setting which is well known as edutainment. As it calls upon the habits in the mass media it is also attractive to parents and kindergarten nurses.

Another program, “Math Kings”, comes form the U.S. It is based on Piaget’s psychology and its extensions by Howard Gardner. I find it quite interesting.

The common feature of all these programs is that they are anchored in pedagogy, psychology or in the teaching practice.

Our “mathe 2000” approach differs fundamentally from them as we consciously take mathematics as the main point of reference. We do not use extrinsic motivation, but rely completely on mathematics.

**T 13** The decisive question, however, is: What is mathematics?

The contemporary answer is: Mathematics is the science of patterns.

We can distinguish different aspects of mathematics:

Mathematics is first and foremost a tool which is used in many fields and at various level by literally everybody.

When we look at the mathematical journals in the libraries mathematics appears as a well organized system of concepts, something which not many people like.

In our view the most important aspect of mathematics is “play”.

**T 14** Perhaps this aspect is new to you and perhaps you have difficulties in relating this aspect to your own mathematical experiences.
To give you an idea how mathematics as play can become vital I would like to invite you to a little exercise with the multiplication table, a piece of mathematics which seems particularly dry.

Imagine you are second graders and have learned the multiplication table to some extent. As you need extensive practice I as your teacher give you an exercise. For this you will need a sheet of paper for your calculations.

In addition I have the so-called broad hundred chart for you.

Now the problem is as follows:

Choose a two digit number. Multiply the tens digit by 3 and add the ones digit. Subtract the result from the chosen number.

If you choose 67 and mark this number on your hundred chart you have to do the following calculation: $6 \cdot 3 + 7 = 25$, $67 - 25 = 42$. Now please write 42 in the right bottom of the box for 67.

If you choose 38 and make a mistake you will get $3 \cdot 3 + 8 = 19$, $38 - 19 = 19$. The final result 19 is again entered into the box for 38, and so on. Similarly other numbers are chosen and we get more results which we enter into our hundred chart.

Now please continue on your own. Choose your own numbers and calculate…

What have you found out?

Yes, all results are multiples of 7. This pattern helps to correct mistakes. Put more precisely: Result = tens digit $\cdot 7$.

Now mathematical play is starting: What happens if we multiply the tens digit by 4 instead of 3, or by 5, or by 9, or by any other one digit number. Please choose one of these numbers and repeat the exercise with the new rule.

If we collect all variations we get a nice super pattern.

There is no end:

You can also multiply the tens digit by 11 and see what happens.

Now let us change the context and let us go back 150 years to a very prominent German pedagogue: Friedrich Froebel. He has become famous for his geometric gifts for small children.

The third gift consists of a cube which is divided in 8 small cubes.
Froebel’s recommendation consists of stimulating children to rearrange the cubes in various ways: decomposing the big cube in halves, in quarters, making real objects: a chair, a sofa, towers, doors and – this is most important – also beautiful patterns beyond any real purpose. It is most interesting to note that Froebel was not only an educator but also a scientist. For some years he worked at the department of mineralogy and crystallography of the University of Berlin.

**T 36** When he declared “play” as a basic category of education, he was clearly influenced by his experiences as a scientist.

The fundamental role of play for children’s development was systematically elaborated 50 years later by G.H. Mead in this book: Mind, Self and Society with an emphasis on play in the social context.

**T 37** At this point I would like to share with you a short section of a clinical interview with two five year old boys. It was conducted in 1986 during a research project on the development of children’s combinatorial thinking. One of the tasks we used was a well-known game of strategy: The race to 10.

**T 38** The rules are as follows: A line of circles is numbered from 1 to 10. The first player starts by putting 1 or 2 counters on the first circle or the first two circles, the second player follows by putting 1 or 2 counters on the next circles similarly. Continuing in this way the players take turns until one of them arrives at the target and in doing so wins the game.

**T 39 - 45** A possible sequence of moves is the following one – the red player wins.

**T 46 - 52** Here is a second sequence - this time the blue player wins.

When playing the game repeatedly children get more and more familiar not only with the number line but also with the mathematical structure of the game.

The language on the video is German. However, there is a lot of nonverbal communication. So you can get at least some impression of the freshness with which the children are acting and you can get even some idea of what is going on in their minds.

((Video 5 minutes))

**T 53** The naïve approach of kindergarten children can be elaborated to a systematic analysis: By pursuing the moves backwards one can classify the positions in positive and negative ones: 10 is a positive position as the player who reaches it wins. 9 and 8 are negative positions as the player who reaches them leaves the opponent the chance to move to the positive position 10. Position 7 is positive as the player who reaches it forces the opponent to move to
a negative position. The same argument shows that 4 and 1 are winning positions. That is, the first player has a winning strategy.

T 54 - 55 As a proud grandfather I remember very well what happened when my wife played the game for the first time with our granddaughter Isabelle, at that time 4 years old. In the third round grandma consciously placed one red counter on position 7 and not two red counters on positions 7 and 8. Isabelle hesitated a moment, looked out of the window and when she returned to the game she said: “Grandma, it’s your turn”.

As our empirical studies showed 4 to 5-year-old children play this game with great pleasure and develop some first insight into the winning strategy. However, this knowledge is fairly instable. A few days later most children have to re-discover what they seemed to have mastered before.

T 56 The understanding of mathematical activity as play forms the right framework for appreciating a statement of one of the great English mathematicians of the 19th century: William Clifford, inventor of the Clifford Algebra which found an application in quantum physics 50 years later.

T 57 What is special about small children?
They are open, curious and active all the time.
In play they repeat and vary actions hundreds of times in order to get better and better results. And most importantly: small children do not know what “mistakes” are – unless adults spoil their natural playful activities.

Our grand son Florian, one of the two twins, is particularly blessed in ignoring failure. He simply doesn’t care if he wins or loses in a game, he even doesn’t ignore it.

T 58 What is the common basis for work in the kindergarten and in the primary school?
In our understanding it consists of fundamental mathematical processes and ideas which are growing steadily in the mind like living organisms.

T 59 We have four fundamental processes:
Mathematizing, searching and problem solving, reasoning, representing and communicating. These are universals from kindergarten to the university.

T 60 - 61 As far as arithmetic is concerned: “mathe 2000” has identified seven fundamental ideas some of which are relevant for the kindergarten.

T 62 - 63 For geometry we have also seven fundamental ideas and some of them are relevant for the kindergarten.
With this mathematical basis in mind we have formulated four principles for early mathematical education:

- We think that work in kindergarten should be delineated from work at school. Learning at school has to be more systematic and more goal directed.
- In the kindergarten children should encounter mathematics by means of mathematical games and construction plans.
- Plays and plans allow for natural repetition and variation which seems to be the most efficient form of learning.
- Adults should not push the children for “output” and should refrain from negative comments.

In the last part of my talk I would like to introduce you into the “mathe 2000”- program for early mathematical education which is based upon these principles.

The program consists of five boxes which contain games and instructions for making things: two boxes deal with numbers, two with geometric forms and one box with logic.

These materials were inspired by several sources:
- Games and construction plans from the folklore around the world
- Froebel’s gifts
- The mirror books by Marion Walter

I will describe some of the activities in detail.

The first activity from vol. 1 of the number book is entitled: Continuing patterns. On the right page you see some linear patterns, at the bottom of both pages you see two dimensional patterns. Children are stimulated to discover the patterns and to continue them.

This activity has been triggered by a section of the talk which Richard Feynman gave when he was awarded the Nobel Prize in Physics in 1965:

I quote from this talk:

> When I was very young – the earliest story I know – when I still ate in a high chair, my father would play a game with me after dinner. He had brought a whole lot of old rectangular bathroom floor tiles from some place in Long Island City. We set them up on end, one next to the other, and I was allowed to push the end one and watch the whole thing to go down. So far so good. Next, the game improved. The tiles were different colors. I must put one white, two blues, one white, two blues, and another
white and then tow blues – I may want to put another blue, but it must be a white. You recognize already the usual insidiousness: first delight him with play, and then slowly inject material of educational value. My mother who is a much more feeling woman began to realize the insidiousness of his efforts and said: “Mel please, let the poor child put a blue tile if he wants to.” My father said; “No, I want him to pay attention to patterns. It is the only thing I can do that is mathematics at this earliest level.

The next game from the second volume of the number book is intended for several players and uses the “insect cards”: 24 cards with 1 to 6 bees, ants, beetles and butterflies.

The game is called “card to card”. It is played on a 4x6 array. The columns are assigned to the numbers 1 to 6. The first player puts one of his card, say 3 bees, in the corresponding column and in doing so reserves this row for the bees. The next player can put 2 bees on the left side or 4 on the right side of the first card. Or he can place 3 ants, or 3 beetles or butterflies above or below the first card. The first card in each row determines the type of the insect for this row. The players take turns and at the end all 24 cards are laid down systematically ordered in a two dimensional array.

My third example is taken from the first volume of the book of forms. It deals with tessellations. We use two basic forms: an isosceles right triangle (semi square) and a rhombus. The smaller side of the triangle has the same length as the side of the rhombus. The angles are 45°, 90° and 135°. The task consists of paving certain figures so that no holes are left. Here we have a house which we can fill as follows:

We could also start differently. Then we would not be able to continue.

Another solution is the following one:

The following game from vol. 2 of the book of forms is again for one person and deals with symmetry. The objective is to build these 16 symmetric houses by fitting 32 halves correctly together. Children must check very carefully if the circular window is small or big, if the door is broad or thin, if the square windows are big or small and if they are placed in the middle of the door or higher.

Folding is a major activity in volume 2 of the book of forms, and of course origami objects are a must here. The most complicated object is this dog.

Climbing, a game of strategy for two players, has been included in the little book of thinking. The rules are as follows. At the beginning a counter is placed on the lowest
position. The playern take turns in moving the counter up to an adjacent position. The player who reaches the top is the winner.

Perhaps you would like to play this game with your neighbour.

T 76 My last example is a revival from the times of the New Math: As material we use animals which are made from legoblocks. As the head, the belly and the tail can be yellow or black there are 8 animals. The name “strummi” is an abbreviation of “structured material”.

We have also road signs which determine who can pass and who can’t.

T 77 Here you see a meadow with two entrances: one allows the access for all animals with 2 yellow parts, the other one excludes animals with a black belly. One animal after the other can be checked if it can enter or not.

T 78 What about the transition from early mathematical education to primary education, that is to the textbook “Das Zahlenbuch”?

T 79 Here is the table of contents of vol. 1 of “Das Zahlenbuch” which shows that the content is subdivided in theme blocks.

T 80 - 81 As far as arithmetic is concerned the first volume of the little book of numbers has been designed as an introduction into the first and third theme block of “Das Zahlenbuch”. Vol. 2 of the little book of numbers is an introduction into the second theme block of “Das Zahlenbuch”. In the same way the activities of the little books of form prepare the geometric themes in “Das Zahlenbuch”. The little book of thinking finds its continuation in the two volumes of the already existing book of thinking.

We are looking forward to the 20th anniversary of “mathe 2000 next May. We are very happy that on this occasion we will be able to offer a coherent, consistent and comprehensive series of materials for early and primary mathematical education. All these materials present mathematics as the science of patterns.